General covariant xp models and the Riemann zeros

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Abstract. We study a general class of models whose classical Hamiltonians are given by $H = U(x)p + V(x)/p$, where x and p are the position and momentum of a particle moving in one dimension, and U and V are positive functions. This class includes the Hamiltonians $H_{\rm I} = x(p + 1/p)$ and $H_{\rm II} = (x + 1/x)(p + 1/p)$, which have been recently discussed in connection with the non trivial zeros of the Riemann zeta function. We show that all these models are covariant under general coordinate transformations. This remarkable property becomes explicit in the Lagrangian formulation which describes a relativistic particle moving in a 1+1 dimensional spacetime whose metric is constructed from the functions U and V. General covariance is maintained by quantization and we find that the spectra are closely related to the geometry of the associated spacetimes. In particular, the Hamiltonian $H_{\rm I}$ corresponds to a flat spacetime, whereas its spectrum approaches the Riemann zeros in average. The latter property also holds for the model $H_{\rm II}$, whose underlying spacetime is asymptotically flat. These results suggest the existence of a Hamiltonian whose underlying spacetime encodes the prime numbers, and whose spectrum provides the Riemann zeros.

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1. Introduction

In 1999 Berry and Keating conjectured that an appropriate quantization of the classical Hamiltonian $H = xp$, of a particle moving on the real line, could provide the long sought spectral realization of the Riemann zeros [1, 2]. These authors were led to this idea by the similarity between the semiclassical spectrum of a regularized version of the xp model and the average distribution of the Riemann zeros. The regularization introduces the constraints $|x| \geq \ell_x$ and $|p| \geq \ell_p$ in position and momentum, such that the product of their minimal values is equal to the Planck constant ($\ell_x \ell_p = 2\pi\hbar$). This proposal was made in the framework of Quantum Chaos and spectral statistics [3, 4, 5]. About the same time, Connes proposed another regularization of xp based on the constraints $|x| \leq \Lambda$ and $|p| \leq \Lambda$, where Λ is a common cutoff [6]. In the limit where Λ is sent to infinity one obtains a continuum spectrum where the Riemann zeros are absortion spectral lines, according to Connes. This interpretation underlies the adelic approach to the Riemann hypothesis. These results have motivated several works in the last years on the xp model, and related quantum mechanical models, for their possible connection with the Riemann zeros [7]-[21] (see [22] for a review on physical approaches to the Riemann hypothesis).

Specially relevant to this paper are the recent works [18, 19], which propose two different modifications of the xp Hamiltonian in order to have bounded classical trajectories and a discrete quantum spectrum. In reference [18], the classical Hamiltonian is $H_{\rm I} = x(p + \ell_p^2/p)$, which adds to xp a non standard term $x\ell_p^2/p$, where ℓ_p is a constant. The latter term implements, in a dynamical way, the constraint $|p| \geq \ell_p$, but one still needs the constraint $x \geq \ell_x$. The classical Hamiltonian $H_{\rm I}$ can be quantized in terms of a self-adjoint operator whose spectrum agrees asymptotically with the first two terms of the Riemann-Mangoldt formula that counts the number of Riemann zeros [23]. The Hamiltonian $H_{\rm I}$ breaks the symmetry between x and p, which is an appealing feature of the xp model. This fact led Berry and Keating to propose a new Hamiltonian $H_{\rm II} = (x + \ell_x^2/x)(p + \ell_p^2/p)$, which restores the $x - p$ symmetry and implements dynamically both constraints on x and p, as can be seen from the appearance of the constants $\ell_{x,p}$ in it.

The aim of this paper is to generalize the previous models, considering Hamiltonians of the form $H = U(x)p + V(x)/p$, where $U(x)$ and $V(x)$ are positive functions defined on intervals of the real line. This class of Hamiltonians have the remarkable property of being general covariant, which means that they maintain their form under general coordinate transformations, i.e. diffeomorphisms $x' = f(x)$. These transformations change the functions U and V, according to prescribed laws, but the physical observables, such as energies, remain unchanged. General covariance is a signature of gauge symmetry, as it occurs in General Relativity. Indeed, we shall show that the present models describe the motion of a relativistic particle moving in a 1+1 dimensional spacetime whose a metric can be constructed in terms of the functions U and V. The classical trajectories of the Hamiltonian $H = U(x)p + V(x)/p$, being

the geodesics of that metric. Hence these generalized xp models acquire a geometrical interpretation which gives new insights into their quantum properties, and in particular their spectrum.

The organization of the paper is as follows. In section 2 we introduce the classical models and show their general covariance. In section 3 we pass from the Hamiltonian to the Lagrangian formulation and present a relativistic spacetime interpretation, which is illustrated with several examples. In section 4 we discuss the classical trajectories in the Hamiltonian and Lagrangian formulations. In section 5 we analyze the semiclassical spectrum of the models introduced in section 3. We quantize the models in section 6 and show that general covariance is maintained. Finally, we present our conclusions. We have included in Appendix A the derivation of the inverse of the semiclassical quantization formula, and in Appendix B the quantization of the Hamiltonian $H = p + \ell_p^2/p$.

2. The classical Hamiltonian

Let us consider a general class of Hamiltonians of the form

$$H = U(x)\, p + \frac{V(x)}{p}, \qquad x \in D, \tag{1}$$

where x and p are the position and momentum of a particle moving in an interval D of the real line, and $U(x)$ and $V(x)$ are positive functions in D. We shall be mainly concerned with intervals that are halflines, $D = (\ell_x, \infty)$, and eventually with segments i.e. $D = (\ell_x, \tilde{\ell}_x)$, The two examples discussed in the introduction correspond to [18, 19]

$$H_{\rm I} \ = x\left(p + \frac{\ell_p^2}{p}\right), \qquad D = (\ell_x, \infty) \tag{2}$$

$$H_{\rm II} = \left(x + \frac{\ell_x^2}{x}\right)\left(p + \frac{\ell_p^2}{p}\right), \qquad D = (0, \infty). \tag{3}$$

Berry and Keating also studied the model (3) on the whole real line, but we shall not consider this case here because the corresponding functions U and V are not positive. The positivity conditions on U and V are necessary, in order to have bounded classical trajectories, but not sufficient, as shown by the example $H = p + \ell_p^2/p$ (see section 3 and Appendix B). It is convenient to write U and V as

$$U(x) = u^2(x), \qquad V(x) = v^2(x), \tag{4}$$

where $u(x)$ and $v(x)$ will also be positive functions. The Hamiltonians (1) change their sign under the time reversal transformation, i.e.

$$x \to x, \quad p \to -p \Longrightarrow H \to -H, \tag{5}$$

which implies that if $\{x(t), p(t)\}$ is a classical trajectory with energy E, so is $\{x(t), -p(t)\}$ with energy $-E$. Upon quantization, the spectrum will contain time

conjugate pairs $\{E_n, -E_n\}$, for appropiate boundary conditions related to the self-adjoint extensions of (1). The breaking of the time reversal symmetry is suggested by the statistical properties of the Riemann zeros, that are described by the Gaussian Unitary Ensemble distribution (GUE) [24, 25].

The Hamiltonian (1) is covariant under general coordinate transformations of the variable x. Indeed, let us consider the infinitesimal canonical transformation

$$x' = x + \epsilon(x), \qquad p' = (1 - \partial_x \epsilon(x))\, p, \qquad |\epsilon(x)| << 1, \tag{6}$$

that preserves the Poisson bracket

$$\{x, p\} = 1 \Longrightarrow \{x', p'\} = 1 + O(\epsilon^2). \tag{7}$$

Substituting these eqs. into (1), one obtains

$$\begin{aligned} H(x,p) = &\ [U(x') - \epsilon(x')\partial_x U(x') + (\partial_{x'}\epsilon(x'))U(x')]\, p' \\ + &\ [V(x') - \epsilon(x')\partial_x V(x') - (\partial_{x'}\epsilon(x'))V(x')]\, \frac{1}{p'} + O(\epsilon^2) \\ = &\ H(x', p') + O(\epsilon^2), \end{aligned} \tag{8}$$

which has the same form as (1), for redefined functions

$$\begin{aligned} U'(x') &= U(x') - \epsilon(x')\partial_x U(x') + (\partial_{x'}\epsilon(x'))U(x'), \\ V'(x') &= V(x') - \epsilon(x')\partial_x V(x') - (\partial_{x'}\epsilon(x'))V(x'). \end{aligned} \tag{9}$$

These equations are the infinitesimal version of the transformation laws of one dimensional tangent and cotangent vectors

$$U'(x') = \left(\frac{dx}{dx'}\right)^{-1} U(x(x')), \qquad V'(x') = \frac{dx}{dx'} V(x(x')). \tag{10}$$

The momentum p transforms also as a cotangent vector (i.e. one form). Another way to state these transformations laws is by saying that the products $U(x)(dx)^{-1}, V(x)dx, pdx$ and also $u(x)(dx)^{-1/2}, v(x)(dx)^{1/2}$ are invariant under reparametrizations of x. To preserve the positivity of the new functions U' and V', we shall restrict ourselves to diffeomorphisms $x' = f(x)$, such that $df(x)/dx > 0$. These diffeormorphisms form the group denoted as Diff$^+$. The interval D is mapped into the new interval $D' = f(D)$. All the models related by diffeomorphisms are equivalent at the classical level. We shall organize them into equivalent classes described by the quotient

$$\mathcal{M}_{\rm cl} = \{U, V, D\}/\text{Diff}^+. \tag{11}$$

Each class can be uniquely characterized by a Hamiltonian which has a particularly simple form,

$$w(x) = U(x) = V(x) \Longrightarrow H = w(x)\left(p + \frac{1}{p}\right). \tag{12}$$

Any other Hamiltonian can be brought into this form by a convenient reparametrization. For example, the models (2) and (3) correspond to

$$H_{\rm I} \;\to w_{\rm I}(x) = x, \qquad\qquad D = (h, \infty), \qquad h = \ell_x \ell_p \tag{13}$$

$$H_{\rm II} \to w_{\rm II}(x) = x + \frac{h^2}{x}, \qquad D = (0, \infty), \qquad h = \ell_x \ell_p \tag{14}$$

where we made the change of variables $x' = \ell_p x$ in both cases. The function $w(x)$ is unique, up to the shift $x \to x + \text{cte}$. We shall call the canonical form (12) the *symmetric gauge*. Other gauges are possible, as for example $U(x) = 1$, which will be briefly discussed at the end of the next section. $w(x)$ is a scalar function that can be computed in any coordinate system as

$$w(x) = u(x)\, v(x). \tag{15}$$

Using the transformation laws of $u(x)$ and $v(x)$ one can verify that $w'(x') = w(x)$, as claimed above. To find the coordinate transformation that brings a model into the symmetric gauge consider the equation

$$\frac{v'(x')}{u'(x')} dx' = \frac{v(x)}{u(x)} dx. \tag{16}$$

In the symmetric gauge $u'(x') = v'(x')$, so integrating (16) yields the mapping $x' = f(x)$, i.e.

$$x' - \ell'_x = f(x) = \int_{\ell_x}^{x} dy\, \frac{v(y)}{u(y)} \tag{17}$$

which is invertible, $x = f^{-1}(x')$, since $v(x)/u(x) > 0$. The constant ℓ'_x is left undetermined by this map, so it can be choosen at will. $w'(x')$ is obtained using eq.(15) and the inverse of (17) as

$$w'(x') = w(x) = u(x)\, v(x) = u(f^{-1}(x'))\, v(f^{-1}(x')). \tag{18}$$

An application of equations (17) and (18), is to show that apparently different models may turn out to be equivalent, as shown by the following case. Consider the model,

$$H_{\rm III} = xp + \ell_x^2\, \frac{p}{x} + \ell_p^2\, \frac{x}{p}, \qquad D = (0, \infty) \tag{19}$$

which, as the model II, is symmetric under the interchange $x - p$, differing from it in the term $\ell_x^2 \ell_p^2 / xp$ in the Hamiltonian. The map (17) becomes in this case

$$x' - \ell'_x = \int_0^x dy\, \frac{\ell_p y}{\sqrt{y^2 + \ell_x^2}} \longrightarrow x' = \ell_p \sqrt{x^2 + \ell_x^2}, \quad \ell'_x = \ell_x \ell_p, \tag{20}$$

which plugged into (18) yields

$$w'_{III}(x') = \ell_p \sqrt{x^2 + \ell_x^2} = x'. \tag{21}$$

so that this model actually coincides with the model I defined in (13).

In the definition of the family of Hamiltonians (1) we have imposed the positivity condition on $U(x)$ and $V(x)$. Let us suppose for a while that $V(x) = 0$. One can see that a reparametrization $x \to x'$ can bring $U(x)$ to x' and so, all the Hamiltonians of the form $U(x)p$, are equivalent to xp.

3. Lagrangian formulation: relativistic spacetime picture

An essential feature of General Relativity is that the fundamental equations of the theory take the same form in all coordinate systems. As shown in the previous section, this is also a feature of the models defined by the Hamiltonians (1), with respect to the coordinate x. Henceforth, one may suspect the existence of a general relativistic theory lying behind the models (1), which would provide them with a spacetime interpretation. In this section we shall show that this is indeed the case via the Lagrangian formulation.

The Lagrangian associated to the Hamiltonian (1) is given by

$$L = p\,\dot{x} - H = p\,\dot{x} - U(x)p - \frac{V(x)}{p}. \tag{22}$$

In standard classical mechanics, the Lagrangian can be expressed solely in terms of x and $\dot{x} = dx/dt$, as $L = m\dot{x}^2/2 - V(x)$, where m is the mass of the particle and $V(x)$ is the potential. To find $L(x, \dot{x})$ in our case, we use the Hamilton equation of motion

$$\dot{x} = \frac{\partial H}{\partial p} = U(x) - \frac{V(x)}{p^2}, \tag{23}$$

to eliminate p in terms of x and $\dot{x}$. This gives two solutions

$$p = \eta\sqrt{\frac{V(x)}{U(x) - \dot{x}}}, \qquad \eta = \pm 1, \tag{24}$$

that depend on the sign of the momenta, $\eta = \operatorname{sign} p$, which is a conserved quantity. The positivity of $U(x)$ and $V(x)$, imply that the velocity $\dot{x}$ must never exceed the value of $U(x)$, for the momentum not to become an imaginary number. Substituting (24) into (22), yields a Lagrangian,

$$L_\eta(x, \dot{x}) = -2\eta\sqrt{V(x)(U(x) - \dot{x})}. \tag{25}$$

for each value of η. Notice that eq.(24) is singular if $V(x) = 0$, so that the Lagrangian cannot be expressed in terms of x and $\dot{x}$. This is precisely the situation of the usual xp Hamiltonian, whose Lagrangian, $L = p(\dot{x} - x)$, has to be considered as a function of the three variables $x, \dot{x}$ and p. Later on we shall give an interpretation of this peculiar fact.

At the classical level we can restrict the motion of the particle to a definite value of η, but not at the quantum level, where both signs would be required. The action S corresponding to (25) is (we choose $\eta = 1$)

$$S = \int dt\, L_1 = -2\int \sqrt{U(x)V(x)\,(dt)^2 - V(x)dt\,dx}, \tag{26}$$

and it coincides with the action of a particle moving in 1+1 dimensional spacetime with metric $g_{\mu\nu}$, i.e

$$S = -\int d\sigma\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}} = -\int \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}, \tag{27}$$

where σ parametrizes the worldline(we have set the mass of the particle to 1). Making the identifications

$$x^0 = t, \qquad x^1 = x, \tag{28}$$

the metric tensor becomes

$$g_{00} = -4U(x)V(x) \equiv -4W(x), \qquad g_{01} = g_{10} = 2V(x), \qquad g_{11} = 0. \tag{29}$$

In our conventions, the square of a line element dx^μ will be defined as

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{30}$$

so that a time-like distance corresponds to $(ds)^2 < 0$, and a space-like distance to $(ds)^2 > 0$. Eq.(29) imply that, under general transformations of the coordinate x, the function $W(x)$ is a scalar, while the $V(x)$ is a cotangent vector, in agreement with the results of section 2. The determinant of the metric (29), given by

$$g \equiv \det\, g_{\mu\nu} = -4V^2(x), \tag{31}$$

implies that $g_{\mu\nu}$ is a non degenerate Minkowski metric since $V(x) > 0$.

To gain further insight into the spacetime structure underlying the xp models, we shall employ the light-cone formalism, which we pass now to describe. Any two dimensional metric is conformally equivalent to a flat metric. This means that it can be written as

$$(ds)^2 = e^{2\chi}\, dx^+ dx^-, \qquad \chi = \chi(x^+, x^-), \tag{32}$$

where $x^\pm$ are the light-cone variables and e^χ is the conformal factor. To find the transformation from the variables x^0, x^1 to the light-cone variables x^+, x^-, we use the transformation law of the metric tensor

$$g'_{\alpha,\beta}(x') = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}\, g_{\mu\nu}(x), \tag{33}$$

where the ligh-cone metric corresponds to

$$g_{++} = g_{--} = 0, \qquad g_{+-} = g_{-+} = \frac{1}{2} e^{2\chi}. \tag{34}$$

Eqs. (29) and (34), allow us to write (33) as

$$\begin{aligned} 0 \;&= \partial_+ x^0 \left[W(x^1)\partial_+ x^0 - V(x^1)\partial_+ x^1 \right] && (35)\\ 0 \;&= \partial_- x^0 \left[W(x^1)\partial_- x^0 - V(x^1)\partial_- x^1 \right] && (36)\\ e^\chi =& -8W(x^1)\partial_+ x^0 \partial_- x^0 + 4V(x^1) \left[\partial_+ x^0 \partial_- x^1 + \partial_+ x^1 \partial_- x^0 \right]. && (37) \end{aligned}$$

Let us suppose, for a while, that x^0 depends non trivially on x^+ and x^-, i.e. $\partial_\pm x^0 \neq 0$. Hence eqs. (35) and (36), would imply

$$\partial_\pm x^0 = \frac{\partial_\pm x^1}{U(x^1)} = \partial_\pm \int_{\ell_x}^{x^1} \frac{dy}{U(y)} \Longrightarrow x^0 = \int_{\ell_x}^{x^1} \frac{dy}{U(y)} + \text{cte}, \tag{38}$$

so that x^0 would be a function of x^1, which is a contradiction since they are independent variables. We shall make the choice that x^0 only depends on x^+

$$x^0 = f(x^+). \tag{39}$$

Eq.(36) is fulfilled automatically, and eq.(35) becomes

$$\partial_+ x^0 = \frac{\partial_+ x^1}{U(x^1)} = \partial_+ \int_{\ell_x}^{x^1} \frac{dy}{U(y)} \Longrightarrow x^0 = \int_{\ell_x}^{x^1} \frac{dy}{U(y)} - g(x^-), \tag{40}$$

and so

$$\int_{\ell_x}^{x^1} \frac{dy}{U(y)} = f(x^+) + g(x^-), \tag{41}$$

where $f(x^+)$ and $g(x^-)$ are generic functions. $x^1(x^+, x^-)$ is given, in an implicit way, by eq.(41). Finally, eqs.(37) and (41) provides the conformal factor,

$$e^{2\chi} = 4W(x^1)\partial_+ f(x^+)\partial_- g(x^-). \tag{42}$$

Equations (39) and (41) give the map from $x^{0,1}$ to $x^\pm$. However the map is not unique due to the freedom in choosing $f(x^+)$ and $g(x^-)$. This simply reflects the invariance of the metric (32) under general conformal transformations, $x^+ \to \tilde{f}(x^+)$ and $x^- \to \tilde{g}(x^-)$.

In the conformal gauge (34), the tensors and connections simplify considerably. The Christoffel symbols, $\Gamma^\lambda_{\mu\nu}$, have only $\pm$ non vanishing components

$$\Gamma^+_{++} = g^{+-}\partial_+ g_{+-} = 2\partial_+\chi, \qquad \Gamma^-_{--} = g^{+-}\partial_- g_{+-} = 2\partial_-\chi, \tag{43}$$

so that the equations of the geodesics $x^\pm(s)$ read

$$\frac{d^2x^\pm}{ds^2} + 2\left(\frac{dx^\pm}{ds}\right)^2 \partial_\pm\chi = 0, \tag{44}$$

where s is the propertime, i.e. $(ds)^2 = -e^{2\chi}dx^+dx^-$. The Ricci tensor, $R_{\mu\nu}$, becomes

$$R_{+-} = R_{-+} = -2\partial_+\partial_-\chi, \qquad R_{++} = R_{--} = 0, \tag{45}$$

and the Ricci scalar, $R = g^{\mu\nu}R_{\mu\nu}$,

$$R = -8e^{-2\chi}\partial_+\partial_-\chi. \tag{46}$$

It is not difficult to show that

$$R(x) = -\frac{1}{V(x)}\partial_x\left[\frac{\partial_x W(x)}{V(x)}\right]. \tag{47}$$

In the symmetric gauge (12), one has $V(x) = w(x)$ and $W(x) = w^2(x)$, so (47) becomes

$$R(x) = -2\frac{\partial_x^2 w(x)}{w(x)} \tag{48}$$

We shall use this formula to relate different xp models to the underlying spacetime geometries.

Flat spacetimes

In flat spacetimes the scalar curvature vanishes. Equation (48) provides the function $w(x)$ corresponding to these cases

$$R(x) = 0, \quad \forall x \in D \Longleftrightarrow w(x) = \alpha x + c, \qquad \alpha \geq 0. \tag{49}$$

The condition $\alpha \geq 0$ comes from the positivity of $w(x)$. If $\alpha > 0$, the shift $x \to x - c/\alpha$, brings $w(x)$ to the form

$$w(x) = \alpha x, \qquad \alpha > 0, \qquad D = (h, \infty). \tag{50}$$

For $\alpha = 1$, this model coincides with (13). The value of $w_0 \equiv w(h) = \alpha h$, is independent of reparametrizations. If $\alpha = 0$, $w(x)$ is constant and the Hamiltonian is simply

$$w(x) = c > 0, \qquad H = c\left(p + \frac{1}{p}\right), \qquad D = (0, \infty), \tag{51}$$

where we have choosen the origin as the boundary of the interval D (see Appendix B).

Berry-Keating model

This model was defined in eq.(14). The spacetime has a scalar curvature

$$w(x) = x + \frac{h^2}{x} \Longrightarrow R(x) = -\frac{4h^2}{x^2(x^2 + h^2)} \tag{52}$$

which is always negative, vanishes asymptotically as x^{-4}, and diverges at the origin as x^{-2}.

Spacetimes with constant negative curvature

Eq.(48) admits a solution with constant negative curvature

$$R(x) = -|R|, \quad \forall x \in D \Longleftrightarrow w(x) = w_0 \cosh(x\sqrt{|R|/2}), \quad w_0 > 0 \tag{53}$$

where $w_0 > 0$ to guarantee the positivity of $w(x)$. We shall take $D = (0, \infty)$. There is also a solution of (48) with positive curvature involving the cosine function, but it requires finite D domains in order to maintain the positivity of $w(x)$. We shall not consider this case below. The interest of solution (53) is that the semiclassical spectrum coincides with that of the harmonic oscillator (see section 5).

Linear-log model

An interesting variation of the linear potential (49) is to add a subleading logarithmic term, i.e.

$$w(x) = \alpha x + \beta \log x \Longrightarrow R(x) = \frac{2\beta}{x^2(\alpha x + \beta \log x)} \to \frac{2\beta}{\alpha x^3}, \tag{54}$$

where the curvature decays asymptotically as x^{-3}, with a sign determined by that of β ($\alpha > 0$).

Power like models

These models are defined by

$$w(x) = A\, x^{\alpha} \quad (A, \alpha > 0) \Longleftrightarrow R(x) = -\frac{2\alpha(\alpha-1)}{x^2}. \tag{55}$$

We impose the condition $\alpha > 0$ to have a monotonic increasing function $w(x)$. If $\alpha \neq 1$, the curvature vanishes asymptotically as x^{-2}, and its sign is negative for $\alpha > 1$ and positive for $0 < \alpha < 1$. In section 5 we shall show that the asymptotic behavior of the curvature is intimately related to the semiclassical spectrum of the model.

3.1. Flat spacetimes

Let us study in more detail the model (50). Since the curvature vanishes, there is a choice of $f(x^+)$ and $g(x^-)$ for which the conformal factor e^{χ} is constant, and therefore the geodesics are straight lines, it is given by

$$f(z) = g(z) = \frac{1}{2\alpha} \log z, \tag{56}$$

which plugged into eqs.(39) and (41) yields ($h \equiv \ell_x$)

$$x^0 = \frac{1}{2\alpha} \log x^+, \qquad x^1 = h(x^+ x^-)^{1/2}, \tag{57}$$

and conformal factor (recall (42))

$$e^{\chi} = h. \tag{58}$$

The line element

$$(ds)^2 = h^2 dx^+ dx^-, \tag{59}$$

implies that the geodesics are straight lines in the $x^{\pm}$ plane. Not the whole $x^{\pm}$ plane is available for the motion of the particle because it is constrainted to the interval $D = (h, \infty)$. In light-cone coordinates the spacetime domain, $\mathcal{U}$, can be obtained from eq.(57)

$$\mathcal{U} = \{(x^+, x^-) \,|\, x^{\pm} \in (0, \infty), \quad x^+ x^- \geq 1\}. \tag{60}$$

If x^+ and x^- denote the vertical and horizontal axes of the plane, then $\mathcal{U}$ is the region in the first quadrant that is above the hyperbola $x^+ x^- = 1$. This hyperbola is the worldline of the point $x^1 = h$. More generally, the worldlines of any point $x^1 \geq h$, are given by the hyperbolas $x^+ x^- = (x^1/h)^2$. x^+ is the light-cone time coordinate and it flows upwards. Eliminating x^+ in eqs.(57) one finds

$$x^1 = h e^{\alpha x^0} (x^-)^{1/2}. \tag{61}$$

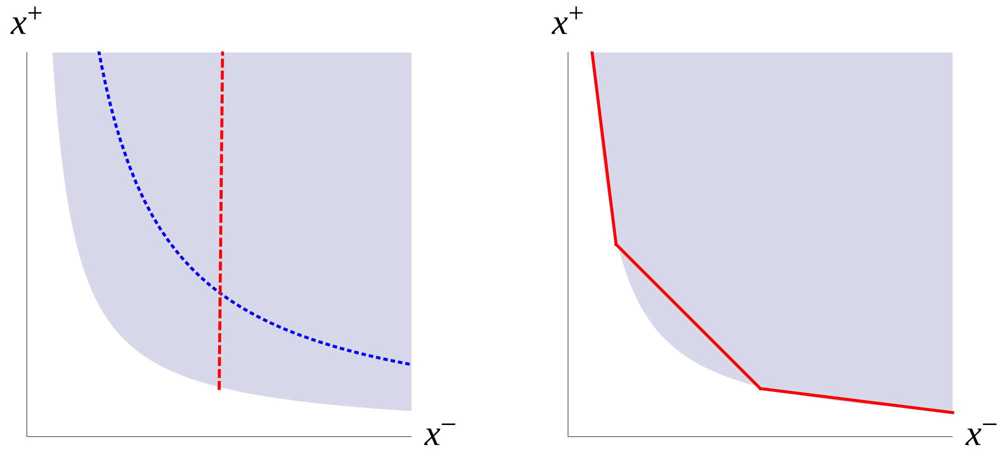

Figure 1. The region in shadow represents the universe $\mathcal{U}$ in light-cone variables defined in eq.(60). Left: the hyperbola (dotted line) corresponds to the worldline of a given position $x^1 = \sqrt{x^+x^-}$ = cte, and the vertical (dashed) line corresponds to a light ray emanating at the boundary (eqs. (61)). Right: the worldline of a particle with constants energy E, which bounces off at the boundary.

Hence, the vertical lines, i.e. constant values of x^-, coincide with the classical solutions of the Hamiltonian, αxp, namely $x \propto e^{\alpha t}$. The line element (59) vanishes along these trajectories, which therefore represent light rays that start at a point on the boundary $x^1 = h$ and scape to infinity as $x^0 \to \infty$ (see fig. 1):

$$x^- = \text{cte} \leftrightarrow \text{light ray}. \tag{62}$$

The line element (59) also vanishes along the horizontal lines, $x^+ =$ cte, but they do not correspond to light rays since the time coordinate x^0 is frozen. In this theory, the light rays are right movers. The left moving light rays are absent. This chirality is a reflection of the time reversal symmetry breaking of the Hamiltonian (1).

The causal cone, i.e. $(ds)^2 < 0$, at each point of $\mathcal{U}$, is given by the second and fourth quadrants, which correspond respectively to the future and past events relative to that point. A particle follows straight lines, with negative slope that start and end at the boundary $x^1 = h$. To show this fact explicitely, we solve the classical equations of motion of the Hamiltonian with $w(x) = \alpha x$, for positive energy E (see section 4)

$$x^2 = \frac{E}{\alpha} e^{2\alpha(t-t_0)} - e^{4\alpha(t-t_0)}, \qquad p^2 = \frac{E}{\alpha} e^{-2\alpha(t-t_0)} - 1, \quad E > 0. \tag{63}$$

In light-cone variables (57) this equation becomes a straight line

$$q^{-\alpha}\, x^+ + q^{\alpha}\, x^- = \frac{E}{w_0}, \qquad E > 0, \quad q = e^{2t_0} h^{1/\alpha} > 0, \quad w_0 = h\alpha, \tag{64}$$

where q parametrizes the slope that depends on the time t_0 where $x(t_0) = |p(t_0)|$. This line ends and starts at the points A and B of the boundary $x^+x^- = 1$, with coordinates

$$x^+_{A,B} = \frac{1}{x^-_{A,B}} = \frac{q^\alpha}{2w_0}\left(E \pm \sqrt{E^2 - 4w_0^2}\right) = q^\alpha\, e^{\pm\alpha\varepsilon}, \tag{65}$$

where

$$\cosh(\alpha\varepsilon) \equiv \frac{E}{2w_0} \geq 1. \tag{66}$$

The energy E of the classical orbits are bounded by $2w_0$. The entire worldline of a particle with energy E, is given by a polygonal line made of linear segments (64), that come from the horizontal axis, $x^0 \to -\infty$, and approaches the vertical axis, $x^0 \to +\infty$ (see fig. 1). The value of q, that parametrizes each segment, can be found matching the initial and final positions of consecutive segments, i.e.

$$x^+_{A_{n-1}} = x^+_{B_n} \to q_n = e^{2\varepsilon} q_{n-1}, \qquad n = -\infty, \infty \tag{67}$$

which means that the particle in the $(n-1)^{\text{th}}$ segment bounces off at $x^1 = h$ and starts a new orbit corresponding to the n^{th} segment. This polygonal worldline represents a periodic motion, since after a shift $x^0 \to x^0 + T_E$, i.e. $x^+ \to e^{2\alpha T_E} x^+$, the equation (64) remains invariant if $q \to e^{2T_E}\, q$ which, according to (67), gives the period T_E as a function of the energy, i.e

$$T_E = \varepsilon = \frac{1}{\alpha} \cosh^{-1} \frac{E}{2w_0}. \tag{68}$$

Let us next study the model defined in equation (51), which also has a vanishing curvature. A choice that leads to a constant conformal factor is

$$f(z) = g(z) = z, \tag{69}$$

which using eqs.(39) and (41) yields

$$x^0 = x^+, \qquad x^1 = c(x^+ + x^-), \tag{70}$$

and

$$e^{\chi} = 2c. \tag{71}$$

The constraint $x^1 \geq 0$ provides the domain of spacetime

$$\mathcal{U} = \{(x^+, x^-) \,|\, x^+ + x^- \geq 0\}. \tag{72}$$

which is depicted in fig.2, which also shows the light-rays and the worldline of the points. The classical equations of motion have the solutions (we choose $E > 0$)

$$x = c(t - t_0)(1 - e^{\varepsilon}), \qquad p = e^{\varepsilon}, \tag{73}$$

$$x = c(t - t_0)(1 - e^{-\varepsilon}), \qquad p = e^{-\varepsilon}, \tag{74}$$

where ε is defined as

$$\cosh\varepsilon = \frac{E}{2c}, \qquad \varepsilon \geq 0. \tag{75}$$

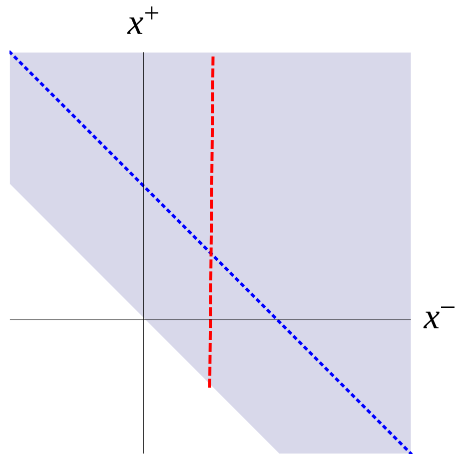

Figure 2. The region in shadow represents the universe $\mathcal{U}$ in lightcone variables defined in eq.(72). Left: the hyperbola (dotted line) corresponds to the world line of a given position $x^1 = c(x^+ + x^-) =$ cte, and the vertical (dashed) line corresponds to a light ray emanating at the boundary $x^+ + x^- = 0$. Right: worldline of a particle with constants energy E, describe by eqs.(76) and (77) with $t_0 = 0$.

The solution (73) describes the particle moving towards the origin, i.e. $\dot{x} < 0$, which it is reached at $t = t_0$. At that moment it bounces off and starts to move to the right, i.e. $\dot{x} > 0$, as described by eq.(74). In the lightcone coordinates (70), eqs.(73, 74) becomes

$$e^{-\varepsilon/2}\, x^+ + e^{\varepsilon/2}\, x^- = -\, 2t_0 \sinh(\varepsilon/2), \tag{76}$$

$$e^{\varepsilon/2}\, x^+ + e^{\varepsilon/2}\, x^- = -\, 2t_0 \sinh(\varepsilon/2). \tag{77}$$

Fig. 2 illustrates the form of these trajectories.

A conclusion of the results we obtained in this subsection is that the scalar curvature by itself does not fully characterize a model and that the boundary of spacetime may play an essential role.

Before we leave this section we want to make some remarks:

- The reformulation of the Hamiltonians (1) as relativistic models described by the Lagrangians (25) holds strictely speaking at the classical level. At the quantum level, this relation is more involved. Indeed, one should start from the path integral in phase space and perform the integration over the variable p. For Hamiltonians of the form $H = p^2/2m + V(x)$, this integral is gaussian and one gets the familiar Feymann path integral with Lagrangian $L = m\dot{x}^2/2 - V(x)$. However for the Hamiltonians (1), the integration over p is not gaussian and in the saddle approximation, one gets extra terms in addition to the Lagrangian (25). In any case, the quantization of these models will be done in section 6 using the Hamiltonian formulation, where this issue does not arise.
- Given a model in the symmetric gauge (12), one can find a new coordinate x' such that $U(x') = 1$, that we shall call the $p-$gauge. The transformation $x' = f(x)$, and

the new function $V'(x') \equiv V_p(x')$, can be derived from eq.(10) and the scalar nature of $W(x)$

$$x' - \ell'_x = f(x) = \int_{\ell_x}^{x} \frac{dy}{w(y)}, \qquad V_p(x') = w^2(f^{-1}(x')). \tag{78}$$

In the $p-$gauge the Hamiltonian takes the canonical form

$$H = p + \frac{V_p(x)}{p}, \tag{79}$$

whose square

$$H^2 = p^2 + 2V_p^2(x) + \frac{V_p^2(x)}{p^2}, \tag{80}$$

coincides with the standard Hamiltonian with a positive potential $2V_p^2(x)$, except for the $V_p^2(x)/p^2$ term. One finds for example that the functions $V_p(x)$ associated to the models (13) and (14) are given by

$$H_{\rm I} \ \rightarrow V_{p,{\rm I}}(x) = h^2\, e^{2x}, \qquad D = (0, \infty), \tag{81}$$

$$H_{\rm II} \rightarrow V_{p,{\rm II}}(x) = \frac{h^2 e^{2x}}{1 - e^{-2x}}, \qquad D = (0, \infty). \tag{82}$$

which seem to have some relation with the Liouville model and the Morse potentials studied in [13] (see references therein), whose spectrum is related to the Riemann zeros in average.

4. Classical trajectories and equations of motion

In the symmetric gauge (12), the classical trajectories in phase space are curves with constant energy E,

$$E = w(x) \left(p + \frac{1}{p} \right). \tag{83}$$

For each position x, there are in general two different values of the momentum p that we denote as

$$p_\pm(x, E) = \frac{1}{2w(x)} \left(E \pm \sqrt{E^2 - 4w^2(x)} \right), \tag{84}$$

and satisfy the relation

$$p_+(x, E) = \frac{1}{p_-(x, E)}. \tag{85}$$

In fig. 3, we plot the classical trajectories corresponding to the models (13) and (14). In all these models, the trajectories are clockwise for positive energy and anticlockwise for negative energy. This is a consequence of the time reversal breaking of the Hamiltonian (1). In the model (13), the particle starts at $x = h$, with a high momentum, say $p_{\rm high}$.

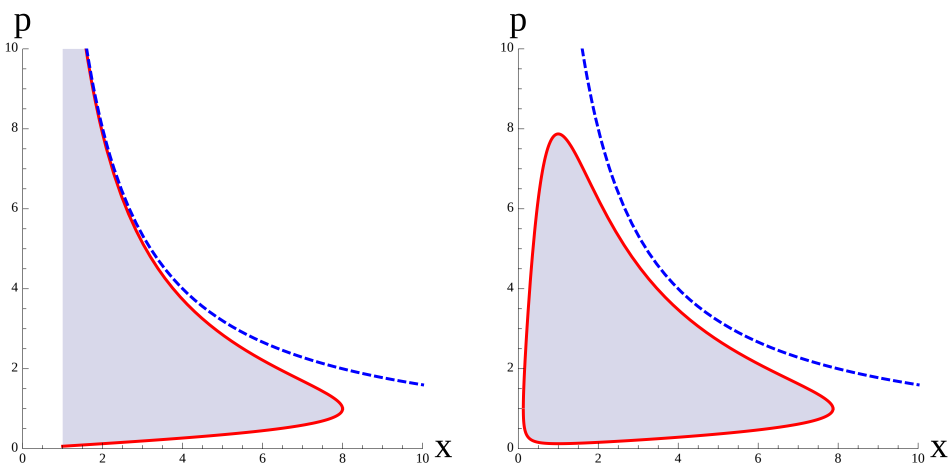

Figure 3. Classical trajectories of the Hamiltonians (2) (left) and (3) (right) in phase space with $E > 0$. We include for comparison the classical trajectory of the xp model, given by the hyperbola $E = xp$ (dashed lines).. The parameters of the latter Hamiltonians are choosen as $\ell_x = \ell_p = 1$. The classical trajectories with negative energy can be obtained from the trajectories with positive energy replacing $p \to -p$.

Then it moves to the right, while its momentum decreases until $|p| = 1$, where x reaches a maximum $x_{\rm M}(E)$. Afterwards, the particle moves back towards lower values of x and $|p|$. When the particle reaches the boundary of the interval, i.e. $x = h$, with a momenta $|p_{\rm low}| < 1$, it bounces off acquiring the original momenta $p_{\rm high} = 1/p_{\rm low}$, that satisfy eq.(85). The latter relation can be written as $\log|p_{\rm high}| = -\log|p_{\rm low}|$ which is the analogue of the reflection of a particle in a wall. After the reflection, the particle restarts the motion from its original position and momentum, so that the classical trajectory is closed and periodic [18].

In the model (14), there is no a wall for the motion of the particle which follows a continuous and differential orbit around the point $(h, 1)$ in phase space [19]. In both models, for large values of x and p, the classical trajectories approach the hyperbola $E = xp$, but they departure from it whenever x or $|p|$ are of the order of the parameters h or 1.

The Hamilton equations of motion for the general Hamiltonian (1) are given by

$$\dot{x} = w(x)\left(1 - \frac{1}{p^2}\right), \qquad \dot{p} = -w'(x)\left(p + \frac{1}{p}\right), \tag{86}$$

where $\dot{x} = dx/dt$, $w'(x) = dw(x)/dx$. Computing $\ddot{x}$, and expressing p as a function of $\dot{x}$ and $w(x)$, one finds

$$\ddot{x} = w(x)w'(x)\left(-4 + \frac{6\dot{x}}{w(x)} - \frac{\dot{x}^2}{w^2(x)}\right). \tag{87}$$

In section 3, we formulated our model as a general relativistic theory. Equations (44) for the geodesics are therefore expected to follow from eq. (87). Indeed, let us choose for simplicity $f(z) = g(z) = z$, so that eqs.(39), (41) and (42) become

$$t = x^+, \qquad \partial_\pm x = w(x), \qquad e^{2\chi} = 4w^2(x), \tag{88}$$

where $x^0 = t$ and $x^1 = x$. The proper time s and the time t along the trajectory are related by

$$(ds)^2 = -e^{2\chi} dx^+ dx^- = 4w^2(x) \left((dt)^2 - \frac{dt\, dx}{w(x)} \right), \tag{89}$$

so

$$\frac{dx^+}{ds} = \frac{1}{2w(x)} \left(1 - \frac{\dot{x}}{w(x)} \right)^{-1/2}. \tag{90}$$

Taking a derivative respect to t in this equation yields

$$\frac{d^2x^+}{ds^2} \div \left(\frac{dx^+}{ds} \right)^2 = \left(1 - \frac{\dot{x}}{w(x)} \right)^{-1} \left[-\frac{w'(x)\dot{x}}{w(x)} + \frac{w'(x)(\dot{x})^2}{2w^2(x)} + \frac{\ddot{x}}{2w(x)} \right] \tag{91}$$

which together with (87), leads to

$$\frac{d^2x^+}{ds^2} + \left(\frac{dx^+}{ds} \right)^2 2w'(x) = 0. \tag{92}$$

This equation coincides with (44) (for x^+) as follows from

$$\partial_+ \chi = \frac{\partial_+ w}{w} = \frac{\partial_+ x \, \partial_x w}{w} = w'(x). \tag{93}$$

The geodesic equation for x^- can be derived in a similar way.

In the model (50) the solution of the eqs. of motion (86) is given by

$$x^2 = \frac{E}{\alpha} e^{2\alpha(t-t_0)} - e^{4\alpha(t-t_0)}, \qquad p^2 = \frac{E}{\alpha} e^{-2\alpha(t-t_0)} - 1, \quad E > 0, \tag{94}$$

where t_0 is the instant where $x = p$. At the initial and final times, $t_{i,f}$ the particle is at $x = h$, which implies

$$e^{2\alpha(t_{f,i}-t_0)} = \frac{E}{2\alpha} \pm \sqrt{\left(\frac{E}{2\alpha} \right)^2 - h^2}, \tag{95}$$

so that the period of the trajectory coincides with (68),

$$T_E = t_f - t_i = \frac{1}{\alpha} \cosh^{-1} \frac{E}{2w_0} \to \frac{1}{\alpha} \log \frac{E}{w_0}, \quad (E >> w_0). \tag{96}$$

The trajectories of the Berry-Keatig model with Hamiltonian $H_{\rm II}$, (recall eq. (14)), are given in terms of elliptic functions [19]. In fig 4, we plot $x(t)$ and $p(t)$ for the Hamiltonians $H_{\rm I}$ and $H_{\rm II}$ with the same energy. The discontinuity of the curve for the $H_{\rm I}$ model is in contrast with its continuity for the $H_{\rm II}$ model. Despite of this fact, both curves have almost the same period. It is remarkable how fast the particle retraces its trajectory near the origin, which is the reason why the periods T_E in the two models converge for large trajectories.

The general expression of the period of a trajectory with energy E, can be obtained integrating eqs.(86)

$$T_E = \int_{x_{\rm m}}^{x_{\rm M}} \frac{dx}{w(x)} \frac{E}{\sqrt{E^2 - 4w^2(x)}}, \qquad E = 2w(x_{\rm m}) = 2w(x_{\rm M}), \tag{97}$$

where x_m and x_M are the turning points. The minimal value x_m can actually coincide with the boundary value of the interval D, as for the $H_{\rm I}$ model.

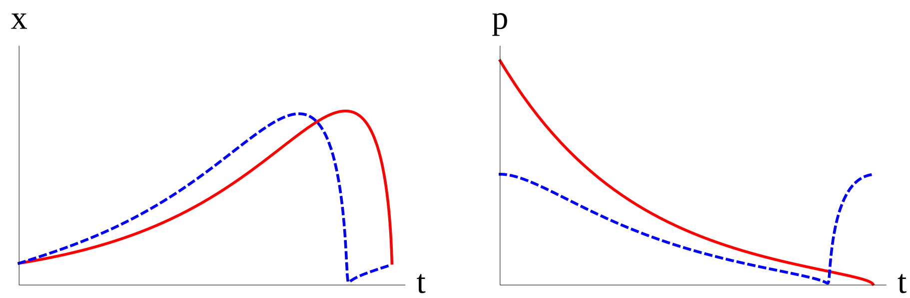

Figure 4. Plot of $x(t)$ and $p(t)$ for the Hamiltonians $H_{\rm I}$ (continuous red curve) and $H_{\rm II}$ (dotted blue curve).

5. Semiclassical analysis

The number of semiclassical states with energy between 0 and $E > 0$, denoted as $n(E)$, is given by the area in phase space swept by the closed trajectory, measured in units of the Planck constant $2\pi\hbar$. In the symmetric gauge it reads

$$n(E) = \frac{1}{2\pi\hbar} \int_{0<H<E} dx\, dp = \frac{1}{2\pi\hbar} \int_{x_{\rm m}}^{x_{\rm M}} \frac{dx}{w(x)} \sqrt{E^2 - 4w^2(x)}, \tag{98}$$

where $x_{m,M}$ are the turning points of the trajectory (recall (97)). We have not included in (98) the Maslow phase. The counting formula for the negative energy states is also given by (98) with $|E| = 2w(x_{m,M})$. In the next section we shall see that this symmery is broken in general by the quantum model. In other gauges, the corresponding formula for $n(E)$ is obtained from (98), by the replacements $dx/w(x) \to dx/U(x)$ and $w^2(x) \to U(x)V(x)$, which shows that $n(E)$ is invariant under reparametrizations of x (see section 2). The derivative of the area of the trajectory, $2\pi\hbar\, n(E)$, respect to E, gives the period (97). In the cases where $w(x)$ is an invertible function, and assuming that $x_{\rm m} = x_0$, one can invert eq.(98) in order to find the function $w(x)$, or rather $x(w)$, that produces a given $n(E)$. The formula is given by (see Appendix A for the proof)

$$\frac{x(w) - x_0}{2\hbar w} = \int_{w_0}^{w} dE\, E \frac{d}{dE} \left(\frac{n(2E)}{E} \right) \frac{1}{\sqrt{w^2 - E^2}}. \tag{99}$$

Let us apply eqs. (98) and (99) to the models studied in section 3.

Linear potential

In the case of the linear potential

$$w(x) = \alpha x, \qquad D = (h, \infty) \tag{100}$$

one finds [18]

$$
\begin{aligned}
n(E) = & \quad \frac{E}{2\pi\hbar\alpha}\left(\cosh^{-1}\frac{E}{2w_0} - \sqrt{1-\left(\frac{2w_0}{E}\right)^2}\right) \\
& \stackrel{E>>1}{\to} \frac{E}{2\pi\hbar\alpha}\left(\log\frac{E}{w_0} - 1\right) + O(E^{-1}) \quad (E >> w_0)
\end{aligned}
\tag{101}
$$

where $w_0 = w(h) = \alpha h$. The leading term, $O(E \log E)$, depends only on α, and the next to leading term, $O(E)$, depends on α and w_0. Let us compare eq.(101), with the Riemann-Mangoldt formula up to a height t [23]

$$
\langle n(t)\rangle \simeq \frac{t}{2\pi}\left(\log\frac{t}{2\pi} - 1\right) + \frac{7}{8} + O(t^{-1}), \qquad t >> 1. \tag{102}
$$

The first two leading terms in this formula agree with those of (101) under the identifications

$$
t = \frac{E}{\hbar\alpha}, \qquad h = 2\pi\hbar. \tag{103}
$$

Setting α to 1, we recover the model I. Notice that the constant term in (101) vanishes, unlike in Riemann's formula where it is given by 7/8.

Linear -log model

One may ask which modification of $w(x)$ would yield a counting function $n(E)$ similar to (101), but containing a constant term, i.e.

$$
n(E) = \frac{E}{2\pi\hbar}\left(\cosh^{-1}\frac{E}{2w_0} - \sqrt{1-\left(\frac{2w_0}{E}\right)^2}\right) + \mu. \tag{104}
$$

To find this potential we use eq.(99)

$$
\begin{aligned}
x - x_0 = & \ w - w_0 + \mu\hbar\log\frac{1-\sqrt{1-(w_0/w)^2}}{1+\sqrt{1-(w_0/w)^2}} \\
& \simeq w - w_0 + 2\mu\hbar\log\frac{w_0}{2w}, \qquad w >> w_0,
\end{aligned}
\tag{105}
$$

whose inverse is

$$
w(x) \sim x + 2\mu\hbar\log x, \qquad x >> 1. \tag{106}
$$

Based on this result we expect that a function $w(x)$ reproducing the Riemann zeros would contain a $\log x$ in its asymptotic expansion.

The Berry-Keating model

The semiclassical spectrum associated to the function $w(x) = x + h^2/x$, defined in the halfline $D = (0, \infty)$, is given by [19]

$$\begin{aligned} n(E) = & \quad \frac{E}{2\pi\hbar}\left[K\left(1-\frac{16h^2}{E^2}\right) - E\left(1-\frac{16h^2}{E^2}\right)\right] \\ & \overset{E>>1}{\to} \frac{E}{2\pi\hbar}\left(\log\frac{E}{h}-1\right) - \frac{2h^2}{\pi\hbar}\frac{\log E}{E} + \ldots, \end{aligned} \tag{107}$$

where K and E are the elliptic integrals

$$K(k^2) = \int_0^{\pi/2} dx\,\left(1-k^2\sin^2 x\right)^{-1/2}, \quad E(k^2) = \int_0^{\pi/2} dx\,\left(1-k^2\sin^2 x\right)^{1/2}. \tag{108}$$

Choosing $t = E/\hbar$ and $h = 2\pi\hbar$, as in (103), one gets an agreement with the first two terms of the Riemann-Mangoldt formula (102), in the large E limit. The constant term is absent in (107) which we believe is due to the absence of the term $\log x$ in $w(x)$, for large values of x. All these features are shared with the linear potential analyzed previously.

Spacetime with constant negative curvature

The semiclassical spectrum associated to the function $w(x)$ given by (53) and a domain $D = (0, \infty)$, is

$$E_n = \hbar\omega(n+\mu), \qquad n(E) = \frac{E}{\hbar\omega} - \mu, \tag{109}$$

where μ and ω are related to the parameters w_0 and the scalar curvature $R = -|R|$ as

$$R = -\frac{1}{2(\mu\hbar)^2}, \qquad \omega = \frac{2w_0}{\mu\hbar}. \tag{110}$$

It is an interesting fact that the harmonic oscillator spectrum is associated with a spacetime with constant negative curvature, at least semiclassically. This result does not contradict the chaotic spectrum associated to models defined in two dimensional euclidean spaces with constant negative curvature, since they have different spatial dimensions, i.v. one versus two [26].

Power like models

To study the semiclassical spectrum of the models defined in (55), we shall distinguish the cases $\alpha < 1$ and $\alpha > 1$, which have very different properties. For functions $w(x)$ that grow slower than x, one obtains

$$0 < \alpha < 1 \Longrightarrow n(E) \propto E^{\frac{1}{\alpha}}, \qquad E_n \propto n^{\alpha} \tag{111}$$

The density of states dn/dE behaves as $E^{\frac{1}{\alpha}-1}$ and diverges in the limit $\alpha \to 0$, where it becomes a continuum. The model $\alpha = 0$ corresponds to a constant $w(x)$ and it has a continuum spectrum (see Appendix A). For potentials that grow faster than x, one finds

$$\alpha > 1 \Longrightarrow n(E) \propto E + O(E^{\frac{1}{\alpha}}), \qquad E_n \propto n + O(n^{\frac{1}{\alpha}}) \tag{112}$$

which is a linear spectrum with corrections. In the limit $\alpha \to \infty$, one recovers the harmonic oscillator spectrum, which is consistent with the exponential growth of $w(x)$ (recall eq.(53)).

In table 1, we collect the results obtained in this section and a comparison with the curvature of the associated spacetime models. From these examples, we draw the following conclusions:

- In models where $w(x)$ becomes linear for $x >> 1$, the first two leading terms of $n(E)$ are of order $E \log E$ and E. The converse is also true, i.e. the latter asymptotic behaviour of $n(E)$ forces $w(x)$ to be linear for $x >> 1$. Correspondingly, the scalar curvature $R(x)$ vanishes faster than x^{-2}.
- The appearance of a constant, in the next leading correction to $n(E)$, requires a logaritmic term in $w(x)$, in addition to the linear term.
- Based on these arguments, one is lead to conjecture that the function $w(x)$, whose quantum spectrum yields the exact Riemann zeros, is of the form

$$w(x) = x + \mu \log x + w_{\rm fl}(x), \tag{113}$$

where $w_{\rm fl}(x)$ represents the fluctuation part of $w(x)$. The role of $w_{\rm fl}(x)$ is to provide the fluctuation term $n_{\rm fl}(t)$ of the Riemann-Mangoldt formula for the exact position of the Riemann zeros,

$$n_R(t) = \langle n(t) \rangle + n_{\rm fl}(t), \qquad n_{\rm fl}(t) = \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + it \right),$$

where $\zeta(s)$ is the Riemann zeta function. It is not clear at the moment how to construct $w_{\rm fl}(x)$, or even if it exists.

Model	$w(x)$	$n(E)$	$R(x)$
Linear potential	x	$\frac{E}{2\pi} \log \frac{E}{2\pi e} + O(E^{-1})$	0
Berry-Keating potential	$x + h^2 x^{-1}$	$\frac{E}{2\pi} \log \frac{E}{2\pi e} + O(\log E/E)$	$-4h^2\, x^{-4}$
Linear-log potential	$x + 2\mu \log x$	$\frac{E}{2\pi} \log \frac{E}{2\pi e} + \mu + O(E^{-1})$	$4\mu x^{-3}$
Harmonic oscillator	$w_0 \cosh(x/2\mu)$	$\mu(E/2w_0 - 1)$	$-1/2\mu^2$
Sublinear potentials	x^α $(0 < \alpha < 1)$	$E^{1/\alpha}$	$-2\alpha(\alpha-1)x^{-2} > 0$
Superlinear potentials	x^α $(\alpha > 1)$	$E + O(E^{1/\alpha})$	$-2\alpha(\alpha-1)x^{-2} < 0$

Table 1.- Semiclassical counting function $n(E)$, and scalar curvature $R(x)$, for the models discussed in the text. We show the asymptotic behaviour of the functions $n(E)$ and $R(x)$, i.e. $E >> 1, x >> 1$, with $\hbar = 1$.

6. Quantization

To quantize the classical Hamiltonian (1) we shall choose the normal ordering prescription

$$\hat{H} = u(x)\,\hat{p}\,u(x) + v(x)\,\frac{1}{\hat{p}}\,v(x), \qquad x \in D = (\ell_x, \infty) \tag{114}$$

where $\hat{p} = -i\hbar d/dx$ and $1/\hat{p}$ is the one dimensional Green function

$$\langle x|\,\frac{1}{\hat{p}}\,|y\rangle = -\frac{i}{\hbar}\,\theta(y-x), \tag{115}$$

written in terms of the Heaviside step function. This normal ordering generalizes the one in references [18, 19]. One can check that $1/\hat{p}$ is the inverse of $\hat{p}$ acting on wavefunctions $\psi(x)$ that vanish in the limit $x \to \infty$,

$$\begin{aligned}\left(\hat{p}\,\frac{1}{\hat{p}}\,\psi\right)(x) &= -\frac{\partial}{\partial x}\int_{\ell_x}^{\infty} dy\,\theta(y-x)\,\psi(y) = \int_{\ell_x}^{\infty} dy\,\delta(y-x)\,\psi(y) = \psi(x),\\ \left(\frac{1}{\hat{p}}\,\hat{p}\,\psi\right)(x) &= -\int_{\ell_x}^{\infty} dy\,\theta(y-x)\,\frac{\partial\psi(y)}{\partial y} = -\lim_{y\to\infty}\psi(y) + \psi(x),\end{aligned} \tag{116}$$

where we have assume that $x > \ell_x$. There exist other possible normal orderings defining $\hat{H}$, as for example $\frac{1}{2}(u^2(x)\,\hat{p} + \hat{p}\,u^2(x) + v^2(x)\,\hat{p}^{-1} + \hat{p}^{-1}\,v^2(x))$ (see also [19]). However, the choice (114) yields a Schroedinger equation which, as we shall see below, is equivalent to a second order differential equation, suplemented with a non local boundary condition. Our construction parallels in that way the Schroedinger equation arising from the quantum Hamiltonian $\hat{H} = \hat{p}^2/2m + V(x)$. The action of (114) on a wavefunction $\psi(x)$ is given by

$$\begin{aligned}(\hat{H}\psi)(x) = &-i\hbar\,u(x)\frac{d}{dx}\,\{u(x)\psi(x)\}\\ &-i\hbar^{-1}\int_{\ell_x}^{\infty} dy\,v(x)\;\theta(y-x)\,v(y)\psi(y).\end{aligned} \tag{117}$$

This operator will have a real spectrum if it is self-adjoint, which requires, first of all, that it be symmetric [28]

$$\langle\psi_1|\hat{H}\psi_2\rangle = \langle\hat{H}\psi_1|\psi_2\rangle, \qquad \forall\psi_1,\psi_2 \in \mathcal{D}(\hat{H}), \tag{118}$$

where $\mathcal{D}(\hat{H})$ is the domain of the operator $\hat{H}$. To study this condition, let us define the quantity [29]

$$\langle\psi_1|\hat{H}\psi_2\rangle - \langle\hat{H}\psi_1|\psi_2\rangle = -\,i\,\Omega_{12}. \tag{119}$$

Using (117) one finds

$$\begin{aligned}\Omega_{12} &= \hbar \int_{\ell_x}^{\infty} dx \frac{d}{dx}(u^2(x)\, \psi_1^*(x)\psi_2(x)) \\ &+ \hbar^{-1} \int_{\ell_x}^{\infty} \int_{\ell_x}^{\infty} dx\, dy\, v(x)v(y)\psi_1^*(x)\psi_2(y)(\theta(x-y)+\theta(y-x)) \\ &= -\hbar u^2(\ell_x)\psi_1^*(\ell_x)\psi_2(\ell_x) + \hbar^{-1} \int_{\ell_x}^{\infty} \int_{\ell_x}^{\infty} dx\, dy\, v(x)v(y)\psi_1^*(x)\psi_2(y),\end{aligned} \tag{120}$$

where we have assumed that $\lim_{x\to\infty} u(x)\,\psi_{1,2}(x) = 0$. Hence $\hat{H}$ is a symmetric operator iff $\Omega_{12} = 0$ which, in view of eq.(120), is guaranteed if $\psi_{1,2}$ satisfy the equation

$$e^{i\vartheta}\, \hbar\, u(\ell_x)\, \psi(\ell_x) + \hbar^{-1} \int_{\ell_x}^{\infty} dx\, v(x)\psi(x) = 0, \tag{121}$$

where ϑ parameterizes the selfadjoint extensions of $\hat{H}$. The Schroedinger equation of the Hamiltonian (117) is given by

$$-i\hbar\, u(x)\frac{d}{dx}\{u(x)\psi(x)\} - i\hbar^{-1} \int_{\ell_x}^{\infty} dy\, v(x)\, \theta(y-x)\, v(y)\psi(y) = E\, \psi(x), \tag{122}$$

which we write as

$$\hbar\, \frac{u(x)}{v(x)}\frac{d}{dx}\{u(x)\psi(x)\} - \frac{iE}{v(x)}\psi(x) + \hbar^{-1} \int_{\ell_x}^{\infty} dy\, \theta(y-x)\, v(y)\psi(y) = 0. \tag{123}$$

Taking a derivative respect to x and letting $x = \ell_x$ one gets

$$\frac{d}{dx}\left[\hbar \frac{u(x)}{v(x)}\frac{d}{dx}\{u(x)\psi(x)\} - \frac{iE}{v(x)}\psi(x)\right] - \hbar^{-1} v(x)\psi(x) = 0, \tag{124}$$

$$\left[\hbar\, \frac{u(x)}{v(x)}\frac{d}{dx}\{u(x)\psi(x)\} - \frac{iE}{v(x)}\psi(x)\right]_{x=\ell_x} + \hbar^{-1} \int_{\ell_x}^{\infty} dy\, v(y)\psi(y) = 0. \tag{125}$$

It would seem that one has to solve simultaneously eqs.(124) and (125). However, for wavefunctions that decay sufficiently fast at infinity, the latter equation follows from the integration of the former one, dropping a term at infinity. Hence, the Schroedinger equation (122) is equivalent to the second order differential equation (124), which justifies the normal ordering (114).

Eqs. (124) and (125) exhibit the general covariance discussed in the section 2. This can be shown using the transformation laws of $u(x), v(x)$ given in section 2, together with that of $\psi(x)$,

$$\psi'(x')\,(dx')^{1/2} = \psi(x)\,(dx)^{1/2}. \tag{126}$$

which preserves the form of the scalar product of the Hilbert space under diffeomorphism,

$$\int_{\ell'_x}^{\infty} dx'\, \left(\psi'(x')\right)^* \psi'(x') = \int_{\ell_x}^{\infty} dx\, \psi^*(x)\, \psi(x). \tag{127}$$

As was done in the classical and semiclassical analysis, it is convenient to work in the symmetric gauge. Let us transform eq.(124) into that gauge. First we write it as

$$\hbar \frac{u(x)}{v(x)} \frac{d}{dx} \left[\hbar \frac{u(x)}{v(x)} \frac{d}{dx} \{u(x)\psi(x)\} - \frac{iE}{u(x)v(x)} u(x)\psi(x) \right] - u(x)\psi(x) = 0,$$

and make the transformation $x \to x'$ (17) to the symmetric gauge (i.e $dx' = dx\, v(x)/u(x)$),

$$\hbar \frac{d}{dx'} \left[\hbar \frac{d\phi(x')}{dx'} - \frac{iE}{w(x')} \phi(x') \right] - \phi(x') = 0, \qquad (128)$$

where $w(x')$ is given by eq.(18), and the function ϕ is defined as

$$\phi(x') = u(x(x'))\, \psi(x(x')). \qquad (129)$$

Notice that $\phi(x)$ is a scalar function, i.e. $\phi'(x') = \phi(x)$. Now we rename x' as z, and write eqs. (128) and (121), for an eigenfunction $\phi_E(z)$, with energy E, as

$$\hbar \frac{d}{dz} \left(\hbar \frac{d\phi_E(z)}{dz} - \frac{iE}{w(z)} \phi_E(z) \right) - \phi_E(z) = 0, \quad z \geq z_0 \qquad (130)$$

$$\hbar\, e^{i\vartheta} \phi_E(z_0) + \hbar^{-1} \int_{z_0}^{\infty} dz\, \phi_E(z) \;= 0. \qquad (131)$$

The norm of the wave function $\psi(x)$ becomes in this gauge

$$\langle \psi | \psi \rangle = \int_{\ell_x}^{\infty} dx\, \psi^*(x)\, \psi(x) = \int_{z_0}^{\infty} \frac{dz}{w(z)}\, \phi^*(z)\, \phi(z). \qquad (132)$$

Let us next discuss the behaviour of the eigenfunctions under the time reversal transformation. The classical model has the property that if $\{x(t), p(t)\}$ is a trajectory with energy E, then $\{x(t), -p(t)\}$ is a trajectory with energy $-E$. Under time reversal, the quantum Hamiltonian (114) changes sign, which leads us to expect a relation between eigenfunctions with energies E and $-E$. Indeed, taking the complex conjugate of eqs.(130,131), and comparing them with those for an eigenfunction $\phi_{-E}(z)$, one finds

$$\phi_E^*(z) \propto \phi_{-E}(z) \Longleftrightarrow e^{i\vartheta} = e^{-i\vartheta} \Longleftrightarrow \vartheta = 0 \text{ or } \pi \qquad (133)$$

Hence, if $\vartheta = 0$ or π, the spectrum displays the time reversal symmetry $E \leftrightarrow -E$ (this fact was already observed in [19]). The difference between these cases resides in the existence of a zero energy state. The solution of equation (130) for $E = 0$ is given by

$$\phi_{E=0}(z) = A\, e^{-z/\hbar} + B\, e^{z/\hbar} \qquad (134)$$

The exponential growing term $e^{z/\hbar}$ will typically yield unnormalizable wave functions, so we only consider the decaying term. In this case the nonlocal boundary condition (131) becomes

$$\hbar(e^{i\vartheta} + 1)e^{-h/\hbar} = 0 \Longrightarrow \theta = \pi \qquad (135)$$

so, only for this value of ϑ is $\phi_0 = e^{-z/\hbar}$ an eigenfunction of the Hamiltonian with a norm given by

$$\langle \psi_0 | \psi_0 \rangle = \int_{z_0}^{\infty} \frac{dz}{w(z)} \, e^{-2z/\hbar} \tag{136}$$

which will be finite for a large class of models that includes the ones studied in sections 3 and 5. In summary, when $\vartheta = 0$, the spectrum of the Hamiltonian (114) contains time reversed pairs $\{E_n, -E_n\}$, excluding the zero energy, while if $\vartheta = \pi$, in addition to the time reversed pairs, there is a normalizable zero energy state. Depending on the form of $w(z)$, the spectrum may, or may not, contain a continuum part.

6.1. The model I

In the following subsection we shall apply the previous formalism to the model with a linear potential $w(x)$ [18]

$$w(z) = z, \qquad D = (z_0, \infty). \tag{137}$$

The eigenfunctions of the Hamiltonian (114) are the solutions of the differential equation (130), which in this case read ($\hbar = 1$)

$$\phi''(z) - \frac{iE}{z} \phi'(z) + \left(\frac{iE}{z^2} - 1 \right) \phi(z) = 0. \tag{138}$$

The solutions of this equation are given essentially by the modified Bessel functions, but only the K-Bessel function gives a normalizable solution

$$\phi_\nu(z) = A_\nu \, z^{1-\nu} \, K_\nu(z), \qquad \nu = \frac{1}{2} - \frac{iE}{2}, \tag{139}$$

where A_ν is the normalization constant. Using eq.(129) one obtains

$$\psi_E(z) = A_\nu \, z^{\frac{iE}{2}} \, K_{\frac{1}{2} - \frac{iE}{2}}(z), \quad z \geq z_0, \tag{140}$$

whose asymptotic behaviour is

$$\psi_E(z) \propto \left\{ \begin{array}{cc} z^{-\frac{1}{2} + iE}, & z_0 < z << E/2 \\ z^{-\frac{1}{2} + \frac{iE}{2}} e^{-z}, & z >> E/2. \end{array} \right. \tag{141}$$

which shows that in the region $z_0 < z << E/2$, the function ψ_E is given approximately by the eigenfunction of the Hamiltonian $\hat{H} = (x\hat{p} + \hat{p}x)/2$ [1, 9, 10]. Notice that $x_M = E/2$ coincides with the maximal elongation of the classical particle. Beyond this value the wave function decays exponentially, as corresponds to the particle entering the classical forbidden region $x > x_{\rm M}$. The equation for the eigenenergies can be obtained plugging (139) into (131), and using the integral [27]

$$\int_{z_0}^{\infty} dz \, z^{1-\nu} \, K_\nu(z) = z_0^{1-\nu} \, K_{\nu-1}(z_0), \tag{142}$$

with the result [18]

$$e^{i\vartheta} K_\nu(z_0) + K_{\nu-1}(z_0) = 0 \to e^{i\vartheta} K_{\frac{1}{2}-\frac{iE}{2}}(z_0) + K_{\frac{1}{2}+\frac{iE}{2}}(z_0) = 0. \tag{143}$$

The asymptotic behaviour of the K-Bessel function

$$K_{\frac{1}{2}+\frac{it}{2}}(z) \sim \sqrt{\frac{\pi}{z}}\, e^{-\pi t/4} \left(\frac{t}{ze}\right)^{it/2}, \quad t >> 1 \tag{144}$$

yields the asymptotic limit of (143)

$$\cos\left(\frac{E}{2}\log\left(\frac{E}{z_0 e}\right) - \frac{\vartheta}{2}\right) = 0, \qquad E >> 1, \tag{145}$$

so that the eigenenergies $E > 0$ behave as

$$n(E) = \frac{E}{2\pi}\left(\log\frac{E}{z_0} - 1\right) - \frac{\vartheta}{2\pi} - \frac{1}{2} \in \mathcal{Z} \tag{146}$$

The two leading terms in this equation agree with the Riemann formula (102) and the semiclassical result (101), with the identification $z_0 = 2\pi$, which is the same as in eq.(103) (notice that $w_0 = z_0 = h$). We must set $\vartheta = 0$ to guarantee time reversed eigenenergies in analogy with the symmetry of the Riemann zeros on the real axis. The quantization of the model brings in a constant factor $-1/2$ in the counting formula (146), so that the factor $7/8$ of Riemann's formula remains unexplained. In this respect, we recall the comment made in the previous section that adding a term $\log x$ to $w(x)$, can give rise to this constant term in the counting formula.

7. Conclusions

In this paper we have studied the properties of a family of one dimensional classical models, and their quantized version, that are extensions of the well known xp model. A fundamental property of these models is that they are covariant under general coordinate transformations, so that they can be organized into equivalent classes that describe the same Physics. This fact suggests some some sort of universality that could perhaps be explained using renormalization group arguments, as the ones employed in reference [8], where the operator $1/\hat{p}$ was also considered.

General covariance manifests itself most clearly in the Lagrangian formulation of a relativistic particle moving in a 1+1 dimensional spacetime. The geometrical properties of this spacetime turn out to be related to the spectral properties of the associated quantum model in an deep way. To wit, when the curvature of spacetime vanishes fast enough at infinity, one obtains a spectrum that coincides with the Riemann zeros on average. It is tempting to think that certain fluctuations of asymptotically flat metrics, may cause fluctuations in the spectrum so as to accurately reproduce the Riemann zeros. It is expected that these fluctuations are determined by the prime numbers, but the precise manner in which this might occur is unclear. This general class of models can

arise as effective descriptions of the dynamics of an electron moving in the plane under the action of a perpendicular uniform magnetic field and a electrostatic potential of the form $V(x,y) = U(x)y + V(x)/y$, where x and y are the two dimensional coordinates [15]. Further investigation will be needed to clarify these issues.

We hope the results presented here shed new light on the spectral interpretation of the Riemann zeros, stimulating the research into such a interdisciplinary topic

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Appendix A. Semiclassical Abel like inversion formula

The aim of this appendix is to derive a formula that, under certain conditions, permits to construct the potential $w(x)$ that reproduces a given semiclassical counting formula $n(E)$. For Hamiltonians of the form $H = p^2/2m + V(x)$, the answer can be obtained using the Abel inversion method [28]. We shall also use Abel's method for Hamiltonians of the form $H = w(x)(p + 1/p)$. First of all, we shall assume that $w(x)$ is a monotonic increasing function in the domain $D = (x_0, \infty)$, so that it is invertible, $x = x(w)$. In this case the semiclassical formula (98) becomes

$$n(E) = \frac{1}{2\pi\hbar}\int_{x_0}^{x_M} \frac{dx}{w(x)}\sqrt{E^2 - 4w^2(x)}, \qquad E = 2w(x_M), \tag{A.1}$$

which we write as

$$\frac{2\pi\hbar\, n(E)}{E} = \int_{x_0}^{x_M} dx \sqrt{\frac{1}{w^2(x)} - \left(\frac{2}{E}\right)^2}. \tag{A.2}$$

Making the change of variables (with $E > 0$)

$$r(x) = \frac{1}{w(x)}, \quad s = \frac{2}{E}, \quad r_0 = \frac{1}{w_0} \geq s,\ w_0 = w(x_0), \tag{A.3}$$

and using the inverse $x(w)$, we transform (A.2) into

$$f(s) \equiv \pi\hbar\, s\, n(s) = \int_{x_0}^{x_M} dx\, \sqrt{r^2(x) - s^2} = \int_{r_0}^{s} dr\, \frac{dx}{dr}\, \sqrt{r^2 - s^2}. \tag{A.4}$$

Differentiating with respect to s yields

$$\frac{df(s)}{ds} = \int_{s}^{r_0} dr\, \frac{dx}{dr}\, \frac{s}{\sqrt{r^2 - s^2}}. \tag{A.5}$$

Next, we use the chain of identities

$$\int_y^{r_0} ds \, \frac{1}{\sqrt{s^2 - y^2}} \, \frac{df(s)}{ds} = \int_y^{r_0} ds \int_s^{r_0} dr \, \frac{dx}{dr} \, \frac{s}{\sqrt{(s^2 - y^2)(r^2 - s^2)}} \tag{A.6}$$
$$= \int_y^{r_0} dr \, \frac{dx}{dr} \int_y^{r} ds \, \frac{s}{\sqrt{(s^2 - y^2)(r^2 - s^2)}} = \frac{\pi}{2} \int_y^{r_0} dr \, \frac{dx}{dr} = \frac{\pi}{2}(x_0 - x(y)),$$

that can be obtained from the Frobenius theorem and the integral

$$\int_a^b dx \frac{x}{\sqrt{(x^2 - a^2)(b^2 - x^2)}} = \frac{\pi}{2}, \qquad 0 < a < b. \tag{A.7}$$

Replacing $f(s) = \pi \hbar s n(s)$ into (A.6), and undoing the change of variables (A.3), yields finally

$$\frac{x(w) - x_0}{2\hbar w} = \int_{w_0}^{w} dE \, E \frac{d}{dE} \left(\frac{n(2E)}{E} \right) \frac{1}{\sqrt{w^2 - E^2}}. \tag{A.8}$$

It is not guaranteed a priori that $x(w)$ is an invertible function. This fact may restrict the functions $n(E)$ that can be obtained as semiclassical spectrum. Another issue concerning (A.8) is the following. Adding to $n(E)$ a term linear in E, produces the same function $w(x)$, as can be easily seen from (A.8). We lost track of this linear term when taking the derivative respect to s in eq.(A.4). To recover this term, one has to plugg the function $w(x)$, obtained from (A.8), back into (A.1), and read the linear term in E.

It is worth to compare these semiclassical formulas with those associated to the standard Hamiltonian $H = p^2 + V(x)$. The analogue of eq. (A.1), for an even potential $V(x)$, is given by

$$n(E) = \frac{2}{\pi \hbar} \int_{x_0}^{x_m} dx \, \sqrt{E - V(x)}, \qquad |E| = V(x_m), \tag{A.9}$$

where we have ignored the Maslow phase, and the analogue of (A.8) is given by

$$x(V) = \hbar \int_{V_0}^{V} dE \, \frac{dn(E)}{dE} \frac{1}{\sqrt{V - E}}. \tag{A.10}$$

The latter equation was used by Wu and Sprung to obtain a potential $V(x)$ whose semiclassical spectrum coincides in average with the Riemann zeros ([31], for a review see [22]),

$$x(V) = \frac{\sqrt{V}}{\pi} \log \left(\frac{2V}{\pi e^2} \right) \Longrightarrow V(x) \propto \left(\frac{x}{\log x} \right)^2, \qquad x, V >> 1 \tag{A.11}$$

This result was used as a seed for a numerical reconstruction of a potential whose spectrum matches a large number of Riemann zeros lying at the bottom part of the critical line. Quite interestingly, that potential has a fractal structure whose dimension is nearby the value 1.5. A problem with this approach is that the Hamiltonian is time

reversal invariant, a fact which does not agree with the distribution of the Riemann zeros which follow the GUE statistic which is characteristic of time reversal breaking random Hamiltonians (see [22]) for a discussion on this issue).

Independently of the previous works, Mussardo employed (A.10) to find a potential whose spectrum behaves, in average, like the prime numbers [32]. The prime number theorem (PNT) states that the number of primes up to x behaves asymptotically as $\pi(x) \sim x/\log x$, so that their density decreases as $d\pi(x)/dx \sim 1/\log x$, as conjectured long ago by Gauss and Legendre [23]. The PNT implies that the $n^{\rm th}$-prime number p_n grows roughly as $p_n \sim n \log n$. Mussardo noticed that this growth allows one to find a quantum mechanical model whose energies are the primes numbers. Choosing the leading term in the expansion of the Riemann formula for $\pi(x)$,

$$\pi(x) \sim Li(x) = \int_2^x \frac{dy}{\log y} \tag{A.12}$$

he obtained

$$x(V) \sim \frac{\sqrt{V}}{\log V} \Longrightarrow V(x) \sim (x \log x)^2, \quad (x, V >> 1) \tag{A.13}$$

This result was also used as a seed for finding a potential whose spectrum matches precisely the lowest prime numbers [22]. As in the case of the Riemann zeros, the prime potential has a fractal structure with dimension near 2. The difference in fractal dimensions, 1.5 versus 2, is consistent with the fact that the Riemann zeros are less random that the primes numbers. The former ones satisfy the GUE statistics and the latter follow an almost Poissonian statistics. It is rather remarkable the *proximity* of the prime number/Riemann zeros potentials to the harmonic oscillator potential. In table 2 we summarize these semiclassical results together with other well known cases.

Model	$V(x)$	$x(V)$	E_n	$n(E)$
Free particle	cte	$0 \leq \lvert x\rvert \leq \infty$	continuum	-
Harmonic oscillator	x^2	$\sqrt{V}$	n	E
Potential well	cte	$0 \leq \lvert x\rvert < 1$	n^2	$\sqrt{E}$
Riemann zeros	$(\frac{x}{\log x})^2$	$\sqrt{V} \log V$	$\frac{2\pi n}{\log n}$	$\frac{E}{2\pi} \log \frac{E}{2\pi e}$
Prime numbers	$(x \log x)^2$	$\frac{\sqrt{V}}{\log V}$	$n \log n$	$\frac{E}{\log E}$

Table 2.- Semiclassical spectrum associated to the classical Hamiltonian $H = p^2 + V(x)$.

Appendix B. Quantization of $H = \hat{p} + \ell_p^2/\hat{p}$

In standard textbooks of Quantum Mechanics it is taught that the momentum operator $\hat{p} = -i\hbar d/dx$ is self-adjoint acting in the Hilbert space of square integrable functions, $L^2(D)$, in two cases: i) $D = \mathbf{R}$ is the real line, and ii) $D = (a, b)$ is a finite interval of the real line [28, 30]. In case i), the operator $\hat{p}$ is essentially self-adjoint, and in case ii) $\hat{p}$ admits infinitely many self-adjoint extensions characterized by the boundary

condition $\psi(b) = e^{i\vartheta}\psi(a)$, where $\vartheta \in [0, 2\pi)$. However, $\hat{p}$ is not self adjoint when D is the halfline, $D = (0, \infty)$, and therefore its spectrum is not real. A *solution* of this problem is suggested by the model we have discussed in this paper. Indeed, let us choose the simplest non vanishing potential $w(x)$, namely a constant

$$w(x) = \ell_p \Longrightarrow \hat{H} = \hat{p} + \frac{\ell_p^2}{\hat{p}}, \tag{B.1}$$

defined on the halfline $D = (0, \infty)$. This model is equivalent to (51), via a scale transformation which gives the relation $c = \ell_p$. We shall show below that $\hat{H}$ is self adjoint and that its spectrum is a continuum and eventually a bound state. This model illustrates in a simple example the more complicated models considered in the main body of the paper. In spite of its simplicity, this model shares several features with the so called Kondo model in Condensed Matter Physics, which suggests that it may have other applications.

To quantize (B.1), we follow the steps of section 6. The Schroedinger equation is given by

$$- i\hbar \frac{d\psi(x)}{dx} - i\frac{\ell_p^2}{\hbar} \int_0^\infty dy\, \vartheta(y - x)\psi(y) = E\psi(x). \tag{B.2}$$

Taking one derivative respect to x, gives

$$- \hbar^2 \frac{d^2\psi(x)}{dx^2} + iE\hbar \frac{d\psi(x)}{dx} + \ell_p^2 \psi(x) = 0, \tag{B.3}$$

whose general solution is

$$\psi_E(x) = A(E)\, e^{ik_+(E)x} + B(E)\, e^{ik_-(E)x}, \tag{B.4}$$

where

$$k_\pm(E) = \frac{1}{2\hbar} \left(E \pm \mathrm{sign}(E)\sqrt{E^2 - 4\ell_p^2} \right). \tag{B.5}$$

We are assuming in (B.5) that E is real, for other values we replace sign(E) by $\pm E/|E|$. Using (B.5) one can compute the von Neumann deffect indices $n_\pm$, which give the number of linearly independent solutions of the equation

$$n_\pm = \dim \{\psi_\pm \,|\hat{H}\, \psi_\pm = \pm iz\, \psi_\pm,\ \ \mathrm{Im}\, z > 0\}. \tag{B.6}$$

Choosing $z = 2\ell_p c$ ($c > 0$), we get two normalizable solutions of (B.5), namely

$$\psi_+(x) = A \exp\left[-\frac{\ell_p\, x}{\hbar} \left(c + \sqrt{c^2 + 1} \right) \right] \Longrightarrow n_+ = 1, \tag{B.7}$$

$$\psi_-(x) = A\mathrm{exp}\left[-\frac{\ell_p\, x}{\hbar} \left(-c + \sqrt{c^2 + 1} \right) \right] \Longrightarrow n_- = 1,$$

hence $n_+ = n_- = 1$, which implies, by the von Neumann theorem, that the operator $\hat{H}$ is self-adjoint, with an infinitely many extensions parametrized by the group $U(1)$

[28, 29]. The spectrum of $\hat{H}$ is given by two intervals whose boundaries are $\pm 2\ell_p$ and $\pm\infty$. and a bound state with eigenvalue E_0

$$\text{spec } \hat{H} = \mathcal{C} \cup \{E_0\} = (-\infty, -2\ell_p) \cup (2\ell_p, \infty) \cup \{E_0\}. \tag{B.8}$$

We denote by $\mathcal{C}$ the continuum part of the spectrum. It is convenient to parametrize the two branches of the continuum as follows

$$E = 2\ell_p \eta \cosh u, \qquad \eta = \text{sign}(E) = \pm 1, \qquad u > 0, \tag{B.9}$$

in which case the momenta (B.5) become

$$k_+(E) = \frac{\eta\, \ell_p}{\hbar} e^{u}, \qquad k_-(E) = \frac{\eta\, \ell_p}{\hbar} e^{-u}, \tag{B.10}$$

and the wave function (B.4)

$$\psi_E(x) = A(E)\, e^{i\eta \ell_p x e^{u}/\hbar} + B(E)\, e^{i\eta \ell_p e^{-u}/\hbar}. \tag{B.11}$$

The discrete eigenvalue of $\hat{H}$ appears for $|E_0| < 2\ell_p$ with a normalizable eigenfunction corresponding to the momenta k_+, i.e.

$$|E_0| < 2\ell_p \Longrightarrow \psi_{E_0}(x) = C\, e^{-k_0 x}, \quad k_0 = -ik_+ = \frac{1}{2\hbar}\left(-iE_0 + \sqrt{4\ell_p^2 - E_0^2}\right). \tag{B.12}$$

To find the value E_0, we impose the non local boundary condition (121), which guarantees that $\hat{H}$ is an hermitean operator,

$$-\, e^{i\vartheta}\psi_{E_0}(0) + \frac{\ell_p}{\hbar}\int_0^\infty dx\, \psi_{E_0}(x) = 0, \tag{B.13}$$

where ϑ is the parameter that characterizes the self-adjoint extension of $\hat{H}$. In eq.(121), we made the shift $\vartheta \to \vartheta + \pi$, so that the first term in (B.13) changed its sign. Plugging (B.12) into (B.13) one finds

$$k_0 = \frac{\ell_p}{\hbar} e^{-i\vartheta}, \tag{B.14}$$

which gives

$$E_0 = 2\ell_p \sin\vartheta, \qquad \vartheta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{B.15}$$

The restriction on ϑ comes from the relation $\cos\vartheta \propto \text{Re}\, k_0 > 0$. If $\pi/2 < \vartheta \leq \pi$, the equation (B.13) is not satisfied and therefore there is no a bound state. We shall restrict below to the case $|\vartheta| < \pi/2$. If $\vartheta = 0$, one gets $E_0 = 0$, so that the spectrum is time reversal symmetric. In the limits $\vartheta \to \pm\pi/2$, one has $E_0 = \pm 2\ell_p$, and $k_0 = \mp i\ell_p/\hbar$, so that the eigenfunction (B.12), becomes a plane wave. The constant C in eq.(B.12), is fixed by the normalization of the wave function

$$\int_0^\infty dx\, |\psi_{E_0}(x)|^2 = 1 \Longrightarrow C = \sqrt{\frac{2\ell_p \cos\vartheta}{\hbar}}. \tag{B.16}$$

The measure of the size of the bound state is given by the average of x,

$$\langle x \rangle = \int_0^\infty dx \, x |\psi_{E_0}(x)|^2 = \frac{\hbar}{2\ell_p \cos\vartheta}, \tag{B.17}$$

and diverges in the limit $\vartheta \to \pm\pi/2$. The operator $\hat{H}$ is self adjoint, then the spectral theorem implies that its eigenfunctions ψ_E form an orthornormal basis, namely

$$\langle \psi_{E_0} | \psi_{E_0} \rangle = \int_0^\infty dx \, \psi_{E_0}(x) \, \psi_{E_0}(x) = 1, \tag{B.18}$$

$$\langle \psi_{E_0} | \psi_E \rangle = \int_0^\infty dx \, \psi^*_{E_0}(x) \, \psi_E(x) = 0, \qquad E \in \mathcal{C}, \tag{B.19}$$

$$\langle \psi_E | \psi_{E'} \rangle = \int_0^\infty dx \, \psi^*_E(x) \, \psi_{E'}(x) = \delta(E - E'), \qquad E, E' \in \mathcal{C}. \tag{B.20}$$

(B.18) coincides with (B.16). Eq. (B.19), gives the relation between the coefficients $A(E)$ and $B(E)$ of the wave function (B.11),

$$\frac{A(E)}{B(E)} = -\frac{k_0^* - i k_+(E)}{k_0^* - i k_-(E)} = -\frac{e^{i\vartheta} - i\eta \, e^{u}}{e^{i\vartheta} - i\eta \, e^{-u}}, \tag{B.21}$$

where we have used eqs.(B.10) and (B.14). Condition (B.20), together with (B.21), fix the form of these coefficients. Using (B.11) one obtains

$$\begin{aligned}
\langle \psi_E | \psi_{E'} \rangle &= A^*_E \, A_{E'} \left[\pi\delta(k'_+ - k_+) + iP \frac{1}{k'_+ - k_+} \right] + B^*_E \, B_{E'} \left[\pi\delta(k'_- - k_-) + iP \frac{1}{k'_- - k_-} \right] \\
&+ A^*_E \, B_{E'} \left[\pi\delta(k'_- - k_+) + iP \frac{1}{k'_- - k_+} \right] + B^*_E \, A_{E'} \left[\pi\delta(k'_+ - k_-) + iP \frac{1}{k'_+ - k_-} \right]
\end{aligned}$$

where $k_\pm = k_\pm(E), k'_\pm = k_\pm(E')$ and $P\frac{1}{x}$ denotes the principal part of $\frac{1}{x}$. To derive this equation we have used the improper integral

$$\int_0^\infty dx \, e^{ikx} = \pi \, \delta(k) + iP \frac{1}{k}, \tag{B.22}$$

which is the integral version of the distribution identity

$$\frac{1}{k + i0} = -i\pi\delta(k) + P\frac{1}{k}. \tag{B.23}$$

Eq.(B.20) is satisfied provided

$$\begin{aligned}
& A^*_E \, A_{E'} \delta(k_+ - k'_+) + B^*_E \, B_{E'} \delta(k_- - k'_-) \\
& + A^*_E \, B_{E'} \delta(k_+ - k'_-) + B^*_E \, A_{E'} \delta(k_- - k'_+) = \frac{1}{\pi} \delta(E - E'),
\end{aligned} \tag{B.24}$$

and

$$(A^*_E, B^*_E) \; M \begin{pmatrix} A_{E'} \\ B_{E'} \end{pmatrix} = 0, \tag{B.25}$$

where M is the matrix

$$M = \begin{pmatrix} \frac{1}{k_+ - k'_+} & \frac{1}{k_+ - k'_-} \\ \frac{1}{k_- - k'_+} & \frac{1}{k_- - k'_-} \end{pmatrix} = \frac{\hbar}{\ell_p} \begin{pmatrix} \frac{1}{\eta\, e^{u} - \eta' e^{u'}} & \frac{1}{\eta\, e^{u} - \eta' e^{-u'}} \\ \frac{1}{\eta\, e^{-u} - \eta' e^{u'}} & \frac{1}{\eta\, e^{-u} - \eta' e^{-u'}} \end{pmatrix}. \qquad \text{(B.26)}$$

Using

$$\delta(k_\pm - k'_\pm) = \frac{\sqrt{E^2 - 4\ell_p^2}}{|k_\pm|}\delta(E - E'), \qquad \delta(k_\pm - k'_\mp) = 0, \qquad \text{(B.27)}$$

eq (B.24) becomes

$$|A_E|^2 \, \frac{\sqrt{E^2 - 4\ell_p^2}}{|k_+|} + |B_E|^2 \, \frac{\sqrt{E^2 - 4\ell_p^2}}{|k_-|} = \frac{1}{\pi}, \qquad \text{(B.28)}$$

and similarly

$$|A_E|^2 \, e^{-u} + |B_E|^2 \, e^{u} = \frac{1}{2\pi\hbar}. \qquad \text{(B.29)}$$

Finally, eq. (B.21) implies

$$\begin{aligned} A(E) &= \frac{e^{i\vartheta} - i\eta\, e^{u}}{\sqrt{8\pi\hbar(\cosh u - \eta \, \sin\vartheta)}}, \\ B(E) &= -\frac{e^{i\vartheta} - i\eta\, e^{-u}}{\sqrt{8\pi\hbar(\cosh u - \eta \, \sin\vartheta)}}. \end{aligned} \qquad \text{(B.30)}$$

These expressions satisfy eq.(B.25), which finally proves that the wave functions (B.11), with coefficients given by (B.30), together with the normalizable state (B.12) form an orthonormal basis. The operator $\hat{H}$, has the physical meaning of momentum rather than energy. Its spectrum (B.8) almost coincide with that of the momentum operator $\hat{p}$ defined on the entire line, except in an interval around the origin $(-2\ell_p, 2\ell_p)$, which is replaced by a bound state localized at the edge of the system. This result is reminiscent of the Kondo model where a bound state is formed between an impurity localized at the origin and the conduction electrons [33]. However in our model there is no spin and it does not describe a many body system, so this analogy is for the time being formal.

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